Four-valued logics of indicative conditionals

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Hence, one ought to distinguish between indicative conditionals and counterfactuals.

We will denote the indicative conditional statement 'if φ then ψ' by $\varphi \to \psi$.

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But what about $\mathbf{1/2} \to x$?

Here, the intuitions may differ...

2. The three-valued case

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	$\neg$
$\mathbf{0}$	$\mathbf{1}$
$\mathbf{1/2}$	$\mathbf{1/2}$
$\mathbf{1}$	$\mathbf{0}$

$\rightarrow_{\mathsf{DF}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$
$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$

$\rightarrow_{\mathsf{OL}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$
$\mathbf{1/2}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$

$\rightarrow_{\mathsf{F}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$
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	$\neg$
0	**1**
1/2	**1/2**
1	**0**

$\rightarrow_{DF}$	**0**	**1/2**	**1**
0	**1/2**	**1/2**	**1/2**
1/2	**1/2**	**1/2**	**1/2**
1	**0**	**1/2**	**1**

$\rightarrow_{OL}$	**0**	**1/2**	**1**
0	**1/2**	**1/2**	**1/2**
1/2	**0**	**1/2**	**1**
1	**0**	**1/2**	**1**

$\rightarrow_{F}$	**0**	**1/2**	**1**
0	**1/2**	**1/2**	**1/2**
1/2	**0**	**1/2**	**1/2**
1	**0**	**1/2**	**1**

$\wedge_{K}$	**0**	**1/2**	**1**
0	**0**	**0**	**0**
1/2	**0**	**1/2**	**1/2**
1	**0**	**1/2**	**1**

$\vee_{K}$	**0**	**1/2**	**1**
0	**0**	**1/2**	**1**
1/2	**1/2**	**1/2**	**1**
1	**1**	**1**	**1**

$\wedge_{OL}$	**0**	**1/2**	**1**
0	**0**	**0**	**0**
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1	**0**	**1**	**1**

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- *Cantwell's logic* CN, induced by $\langle \mathbf{CN_3}, \{\mathbf{1/2}, \mathbf{1}\}\rangle$, with $\mathbf{CN_3} := \langle A_3; \neg, \wedge_{\mathsf{K}}, \vee_{\mathsf{K}}, \rightarrow_{\mathsf{OL}}\rangle$.

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These results can be achieved for DF, OL, F and CN.

In short, we wish to identify:

$$\mathbf{0} \mapsto (0,1), \quad \mathbf{1/2} \mapsto (1,1), \quad \mathbf{1} \mapsto (1,0).$$

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Let us denote $\top := {}^{1}\!/_{2}$ in what follows.

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Let us denote $\top := \mathbf{1/2}$ in what follows. What if we send:

$$\perp \mapsto (0, 0)$$

and compute the tables?

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The tables look as follows:

$\to_{DF}$	$\mathbf{0}$	$\perp$	$\top$	$\mathbf{1}$
$\mathbf{0}$	$\top$	$\top$	$\top$	$\top$
$\perp$	$\mathbf{0}$	$\perp$	$\mathbf{0}$	$\perp$
$\top$	$\top$	$\top$	$\top$	$\top$
$\mathbf{1}$	$\mathbf{0}$	$\perp$	$\top$	$\mathbf{1}$

$\to_{OL}$	$\mathbf{0}$	$\perp$	$\top$	$\mathbf{1}$
$\mathbf{0}$	$\top$	$\top$	$\top$	$\top$
$\perp$	$\top$	$\top$	$\top$	$\top$
$\top$	$\mathbf{0}$	$\perp$	$\top$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$\perp$	$\top$	$\mathbf{1}$

$\to_{F}$	$\mathbf{0}$	$\perp$	$\top$	$\mathbf{1}$
$\mathbf{0}$	$\top$	$\top$	$\top$	$\top$
$\perp$	$\top$	$\mathbf{1}$	$\top$	$\mathbf{1}$
$\top$	$\mathbf{0}$	$\mathbf{0}$	$\top$	$\top$
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The tables look as follows:

$\to_{\mathsf{DF}}$	$\mathbf{0}$	$\bot$	$\top$	$\mathbf{1}$
$\mathbf{0}$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\mathbf{0}$	$\bot$	$\mathbf{0}$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$
$\mathbf{1}$	$\mathbf{0}$	$\bot$	$\top$	$\mathbf{1}$

$\to_{\mathsf{OL}}$	$\mathbf{0}$	$\bot$	$\top$	$\mathbf{1}$
$\mathbf{0}$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$
$\top$	$\mathbf{0}$	$\bot$	$\top$	$\mathbf{1}$
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$\to_{\mathsf{F}}$	$\mathbf{0}$	$\bot$	$\top$	$\mathbf{1}$
$\mathbf{0}$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\mathbf{1}$	$\top$	$\mathbf{1}$
$\top$	$\mathbf{0}$	$\mathbf{0}$	$\top$	$\top$
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In addition,

	$\neg$
$\mathbf{0}$	$\mathbf{1}$
$\bot$	$\bot$
$\top$	$\top$
$\mathbf{1}$	$\mathbf{0}$

$\wedge_{\mathsf{K}}$	$\mathbf{0}$	$\bot$	$\top$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\bot$	$\mathbf{0}$	$\top$	$\mathbf{0}$	$\top$
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$\top$	$\mathbf{0}$	$\bot$	$\top$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$\bot$	$\mathbf{1}$	$\mathbf{1}$

Where one sets

$$x \vee_{\mathsf{K}} y := \neg(\neg x \wedge_{\mathsf{K}} \neg y),$$

$$x \vee_{\mathsf{OL}} y := \neg(\neg x \wedge_{\mathsf{OL}} \neg y).$$

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Let $A_4 := \{\mathbf{0}, \perp, \top, \mathbf{1}\}$. Our four-valued logics are:

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Then, it turns out that one can prove twist representation results for $\mathrm{DFg}, \mathrm{CNg}$ and Fg.

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This entails a different behavior with respect to negation. We wish to define:

$$-(x, y) := (\neg x, \neg y).$$

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In the previous approach, one could see $\top$ and $\bot$ as corresponding to semantic *gluts* and *gaps*, respectively.

However, one could see $\top$ and $\bot$ as representing *vacuously true* and *vacuously false* statements.

This entails a different behavior with respect to negation. We wish to define:

$$-(x, y) := (\neg x, \neg y).$$

Hence, we extend the tables above with:

	$-$
$\mathbf{0}$	$\mathbf{1}$
$\bot$	$\top$
$\top$	$\bot$
$\mathbf{1}$	$\mathbf{0}$

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Then, one can prove twist representation results for DFf^- and Ff.

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iv. Provide a more systematic philosophical account for the new logics.

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